

Weak type estimates for singular integral operators with variable kernels on the weighted Hardy spaces

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Abstract

Let T_Ω be the singular integral operator with variable kernel $\Omega(x, z)$. In this paper, by using the atomic decomposition theory of weighted Hardy spaces, we will obtain the weighted weak type estimates of T_Ω on these spaces, under some Dini type conditions imposed on the kernel $\Omega(x, z)$. This result is new even in the unweighted case.

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1 Introduction

Let S^{n-1} be the unit sphere in \mathbb{R}^n ($n \geq 2$) equipped with the normalized Lebesgue measure $d\sigma$. A function $\Omega(x, z)$ defined on $\mathbb{R}^n \times \mathbb{R}^n$ is said to belong to $L^\infty(\mathbb{R}^n) \times L^r(S^{n-1})$, $r \geq 1$, if it satisfies the following conditions:

- (1) for all $\lambda > 0$ and $x, z \in \mathbb{R}^n$, $\Omega(x, \lambda z) = \Omega(x, z)$;
- (2) for all $x \in \mathbb{R}^n$, $\int_{S^{n-1}} \Omega(x, z') d\sigma(z') = 0$;
- (3) $\|\Omega\|_{L^\infty(\mathbb{R}^n) \times L^r(S^{n-1})} := \sup_{x \in \mathbb{R}^n} \left(\int_{S^{n-1}} |\Omega(x, z')|^r d\sigma(z') \right)^{1/r} < \infty$,

where $z' = z/|z|$ for any $z \in \mathbb{R}^n \setminus \{0\}$. Set $K(x, z) = \frac{\Omega(x, z')}{|z|^n}$. In this paper, we consider the singular integral operator with variable kernel defined by

$$T_\Omega f(x) = \text{P.V.} \int_{\mathbb{R}^n} K(x, x-y) f(y) dy.$$

In [2] and [3], Calderón and Zygmund investigated the L^p boundedness of singular integral operators with variable kernels. They found that these operators are closely related to the problem about second order elliptic partial differential equations with variable coefficients. We will denote the conjugate exponent of $p > 1$ by $p' = p/(p-1)$. In [4], Calderón and Zygmund proved the following theorem.

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Theorem A ([4]). *Let $1 < p, r < \infty$ satisfy*

- (i) $\frac{1}{r} < \frac{1}{p'} + \frac{1}{p'(n-1)}$ *if $1 < p \leq 2$; or*
- (ii) $\frac{1}{r} < \frac{1}{p} + \frac{1}{p(n-1)}$ *if $2 \leq p < \infty$.*

Suppose that $\Omega(x, z) \in L^\infty(\mathbb{R}^n) \times L^r(S^{n-1})$. Then there exists a constant $C > 0$ independent of f such that

$$\|T_\Omega(f)\|_{L^p} \leq C\|f\|_{L^p}.$$

In particular, T_Ω is bounded on $L^p(\mathbb{R}^n)$ for all $p \geq r'$.

In 2008, Lee et al.[13] considered the weighted case and showed that if the kernel $K(x, y)$ satisfies the L^r -Hörmander condition with respect to x and y variables respectively, then T_Ω is bounded on $L_w^p(\mathbb{R}^n)$. More precisely, they proved

Theorem B ([13]). *Let $1 < r < \infty$. Suppose that $\Omega(x, z) \in L^\infty(\mathbb{R}^n) \times L^r(S^{n-1})$ such that the following two inequalities*

$$\sup_{\substack{x \in \mathbb{R}^n \\ 0 < |y| < R}} \sum_{k=1}^{\infty} (2^k R)^{n/r'} \left(\int_{2^k R \leq |z| < 2^{k+1} R} |K(x, z-y) - K(x, z)|^r dz \right)^{1/r} < \infty \quad (1.1)$$

and

$$\sup_{\substack{x, y \in \mathbb{R}^n \\ 0 < |x-y| < R}} \sum_{k=1}^{\infty} (2^k R)^{n/r'} \left(\int_{2^k R \leq |z| < 2^{k+1} R} |K(x, z) - K(y, z)|^r dz \right)^{1/r} < \infty \quad (1.2)$$

hold for all $R > 0$. If $r' \leq p < \infty$ and $w \in A_{p/r'}$, then T_Ω is bounded on $L_w^p(\mathbb{R}^n)$.

In [5, 6], Ding et al. introduced some definitions about the variable kernel $\Omega(x, z)$ when they studied the H^1 - L^1 boundedness of Marcinkiewicz integral. Replacing the above condition (3), they strengthened it to the condition

$$(3') \sup_{\substack{x \in \mathbb{R}^n \\ \rho \geq 0}} \left(\int_{S^{n-1}} |\Omega(x + \rho z', z')|^r d\sigma(z') \right)^{1/r} < \infty.$$

For $r \geq 1$, a function $\Omega(x, z)$ is said to satisfy the L^r -Dini condition if the conditions (1), (2), (3') hold and

$$\int_0^1 \frac{\omega_r(\delta)}{\delta} d\delta < \infty, \quad (1.3)$$

where

$$\omega_r(\delta) := \sup_{\substack{x \in \mathbb{R}^n \\ \rho \geq 0}} \left(\int_{S^{n-1}} \sup_{\substack{y' \in S^{n-1} \\ |y' - z'| \leq \delta}} |\Omega(x + \rho z', y') - \Omega(x + \rho z', z')|^r d\sigma(z') \right)^{1/r}.$$

In order to obtain the H_w^p - L_w^p boundedness of T_Ω , Lee et al.[13] generalized the L^r -Dini condition by replacing (1.3) to the following condition(see also [14])

$$\int_0^1 \frac{\omega_r(\delta)}{\delta^{1+\alpha}} d\delta < \infty, \quad 0 \leq \alpha \leq 1. \quad (1.4)$$

If Ω satisfies (1.4) for some $r \geq 1$ and $0 \leq \alpha \leq 1$, we call it satisfies the $L^{r,\alpha}$ -Dini condition. For the special case $\alpha = 0$, it reduces to the L^r -Dini condition. For $0 < \beta < \alpha \leq 1$, if Ω satisfies the $L^{r,\alpha}$ -Dini condition, then it also satisfies the $L^{r,\beta}$ -Dini condition. We thus denote by Din_α^r the class of all functions which satisfy the $L^{r,\beta}$ -Dini condition for all $0 < \beta < \alpha$.

Theorem C ([13]). *Let $0 < \alpha \leq 1$ and $n/(n + \alpha) < p < 1$. Suppose $\Omega \in \text{Din}_\alpha^r$ such that (1.1) and (1.2) hold for a certain large number r . If $w^{r'} \in A_{(p+\frac{p\alpha}{n}-\frac{1}{r})r'}$, then there exists a constant $C > 0$ independent of f such that*

$$\|T_\Omega(f)\|_{L_w^p} \leq C\|f\|_{H_w^p}.$$

The main purpose of this article is to study the weak type estimates of T_Ω on the weighted Hardy spaces $H_w^p(\mathbb{R}^n)$ at the endpoint case of $p = n/(n + \alpha)$. We now present our main result as follows.

Theorem 1.1. *Let $0 < \alpha < 1$, $p = n/(n + \alpha)$, $r > 1$ and $w^{r'} \in A_1$. Suppose $\Omega \in \text{Din}_\alpha^r$ such that (1.1) and (1.2) hold. Then there exists a constant $C > 0$ independent of f such that*

$$\|T_\Omega(f)\|_{WL_w^p} \leq C\|f\|_{H_w^p}.$$

In particular, if we take w to be a constant function, then we can get

Corollary 1.2. *Let $0 < \alpha < 1$, $p = n/(n + \alpha)$ and $r > 1$. Suppose $\Omega \in \text{Din}_\alpha^r$ such that (1.1) and (1.2) hold. Then there exists a constant $C > 0$ independent of f such that*

$$\|T_\Omega(f)\|_{WL^p} \leq C\|f\|_{H^p}.$$

2 Notations and preliminaries

The definition of A_p class was first used by Muckenhoupt [16], Hunt, Muckenhoupt and Wheeden [9], and Coifman and Fefferman [1] in the study of weighted L^p boundedness of Hardy-Littlewood maximal functions and singular integrals. Let w be a nonnegative, locally integrable function defined on \mathbb{R}^n ; all cubes are assumed to have their sides parallel to the coordinate axes. We say that $w \in A_p$, $1 < p < \infty$, if

$$\left(\frac{1}{|Q|} \int_Q w(x) dx \right) \left(\frac{1}{|Q|} \int_Q w(x)^{-1/(p-1)} dx \right)^{p-1} \leq C \quad \text{for every cube } Q \subseteq \mathbb{R}^n,$$

where C is a positive constant which is independent of the choice of Q .

For the case $p = 1$, $w \in A_1$, if

$$\frac{1}{|Q|} \int_Q w(x) dx \leq C \cdot \text{ess inf}_{x \in Q} w(x) \quad \text{for every cube } Q \subseteq \mathbb{R}^n.$$

A weight function w is said to belong to the reverse Hölder class RH_s if there exist two constants $s > 1$ and $C > 0$ such that the following reverse Hölder inequality holds

$$\left(\frac{1}{|Q|} \int_Q w(x)^s dx \right)^{1/s} \leq C \left(\frac{1}{|Q|} \int_Q w(x) dx \right) \quad \text{for every cube } Q \subseteq \mathbb{R}^n.$$

It is well known that if $w \in A_p$ with $1 < p < \infty$, then $w \in A_r$ for all $r > p$, and $w \in A_q$ for some $1 < q < p$. We thus write $q_w \equiv \inf\{q > 1 : w \in A_q\}$ to denote the critical index of w . Given a cube Q and $\lambda > 0$, λQ denotes the cube with the same center as Q whose side length is λ times that of Q . $Q = Q(x_0, l)$ denotes the cube centered at x_0 with side length l . For a weight function w and a measurable set E , we denote the Lebesgue measure of E by $|E|$ and set the weighted measure $w(E) = \int_E w(x) dx$.

We give the following results that will be used in the sequel.

Lemma 2.1 ([8]). *Let $w \in A_1$. Then, for any cube Q and any $\lambda > 1$, there exists an absolute constant $C > 0$ such that*

$$w(\lambda Q) \leq C \cdot \lambda^n w(Q),$$

where C does not depend on Q nor on λ .

Lemma 2.2 ([10]). *Let $s > 1$ and $A_1^s = \{w : w^s \in A_1\}$. Then we have*

$$A_1^s = A_1 \cap RH_s.$$

Lemma 2.3 ([8]). *Let $w \in A_1$. Then there exists a constant $C > 0$ such that*

$$C \cdot \frac{|E|}{|Q|} \leq \frac{w(E)}{w(Q)}$$

for any measurable subset E of a cube Q .

Given a weight function w on \mathbb{R}^n , for $0 < p < \infty$, we denote by $L_w^p(\mathbb{R}^n)$ the space of all functions satisfying

$$\|f\|_{L_w^p(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |f(x)|^p w(x) dx \right)^{1/p} < \infty.$$

We also denote by $WL_w^p(\mathbb{R}^n)$ the weighted weak L^p space which is formed by all functions satisfying

$$\|f\|_{WL_w^p(\mathbb{R}^n)} = \sup_{\lambda > 0} \lambda \cdot w(\{x \in \mathbb{R}^n : |f(x)| > \lambda\})^{1/p} < \infty.$$

We write $\mathcal{S}(\mathbb{R}^n)$ to denote the Schwartz space of all rapidly decreasing infinitely differentiable functions and $\mathcal{S}'(\mathbb{R}^n)$ to denote the space of all tempered distributions, i.e., the topological dual of $\mathcal{S}(\mathbb{R}^n)$. For any $0 < p \leq 1$, the

weighted Hardy spaces $H_w^p(\mathbb{R}^n)$ can be defined in terms of maximal functions. Let φ be a function in $\mathcal{S}(\mathbb{R}^n)$ satisfying $\int_{\mathbb{R}^n} \varphi(x) dx = 1$. Set

$$\varphi_t(x) = t^{-n} \varphi(x/t), \quad t > 0, x \in \mathbb{R}^n.$$

We will define the maximal function $M_\varphi f(x)$ by

$$M_\varphi f(x) = \sup_{t>0} |(\varphi_t * f)(x)|.$$

Then $H_w^p(\mathbb{R}^n)$ consists of those tempered distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ for which $M_\varphi f \in L_w^p(\mathbb{R}^n)$ with $\|f\|_{H_w^p} = \|M_\varphi f\|_{L_w^p}$. The real-variable theory of weighted Hardy spaces has been investigated by many authors. For example, Garcia-Cuerva [7] studied the atomic decomposition and the dual spaces of H_w^p for $0 < p \leq 1$. The molecular characterization of H_w^p for $0 < p \leq 1$ was given by Lee and Lin [12]. We refer the readers to [7, 12, 18] and the references therein for further details.

In this article, we will use Garcia-Cuerva's atomic decomposition theory for weighted Hardy spaces in [7, 18]. We characterize weighted Hardy spaces in terms of atoms in the following way.

Let $0 < p \leq 1 \leq q \leq \infty$ and $p \neq q$ such that $w \in A_q$ with critical index q_w . Set $[\cdot]$ the greatest integer function. For $s \in \mathbb{Z}_+$ satisfying $s \geq N = [n(q_w/p - 1)]$, a real-valued function $a(x)$ is called a (p, q, s) -atom centered at x_0 with respect to w (or a w -(p, q, s)-atom centered at x_0) if the following conditions are satisfied:

- (a) $a \in L_w^q(\mathbb{R}^n)$ and is supported in a cube Q centered at x_0 ;
- (b) $\|a\|_{L_w^q} \leq w(Q)^{1/q-1/p}$;
- (c) $\int_{\mathbb{R}^n} a(x) x^\alpha dx = 0$ for every multi-index α with $|\alpha| \leq s$.

Theorem 2.4. *Let $0 < p \leq 1 \leq q \leq \infty$ and $p \neq q$ such that $w \in A_q$ with critical index q_w . For each $f \in H_w^p(\mathbb{R}^n)$, there exist a sequence $\{a_j\}$ of w -(p, q, N)-atoms and a sequence $\{\lambda_j\}$ of real numbers with $\sum_j |\lambda_j|^p \leq C \|f\|_{H_w^p}^p$ such that $f = \sum_j \lambda_j a_j$ both in the sense of distributions and in the H_w^p norm.*

In particular, for w equals to a constant function, we shall denote $WL_w^p(\mathbb{R}^n)$ and $H_w^p(\mathbb{R}^n)$ simply by $WL^p(\mathbb{R}^n)$ and $H^p(\mathbb{R}^n)$.

Throughout this article C denotes a positive constant, which is independent of the main parameters and not necessarily the same at each occurrence.

3 Proof of Theorem 1.1

In order to prove our main theorem, we shall need the following superposition principle on weighted weak type estimates.

Lemma 3.1. *Let $w \in A_1$ and $0 < p < 1$. If a sequence of measurable functions $\{f_j\}$ satisfy*

$$w(\{x \in \mathbb{R}^n : |f_j(x)| > \alpha\}) \leq \alpha^{-p} \quad \text{for all } j \in \mathbb{Z}$$

and

$$\sum_{j \in \mathbb{Z}} |\lambda_j|^p \leq 1,$$

then we have that $\sum_j \lambda_j f_j(x)$ is absolutely convergent almost everywhere and

$$w\left(\left\{x \in \mathbb{R}^n : \left|\sum_j \lambda_j f_j(x)\right| > \alpha\right\}\right) \leq \frac{2-p}{1-p} \cdot \alpha^{-p}.$$

Proof. The proof of this lemma is similar to the corresponding result for the unweighted case which can be found in [17]. See also [15, p. 123]. \square

Following the same arguments as in the proof of Lemma 5 in [11], we can also establish the following lemma on the variable kernel Ω . See [6] and [13].

Lemma 3.2. *Let $r \geq 1$. Suppose that $\Omega(x, z) \in L^\infty(\mathbb{R}^n) \times L^r(S^{n-1})$ satisfies the L^r -Dini condition in Section 1. If there exists a constant $0 < \gamma \leq \frac{1}{2}$ such that $|y| < \gamma R$, then for any $x_0 \in \mathbb{R}^n$, we have*

$$\begin{aligned} & \left(\int_{R \leq |x| < 2R} |K(x + x_0, x - y) - K(x + x_0, x)|^r dx \right)^{1/r} \\ & \leq C \cdot R^{-n/r'} \left(\frac{|y|}{R} + \int_{|y|/2R}^{|y|/R} \frac{\omega_r(\delta)}{\delta} d\delta \right), \end{aligned}$$

where the constant $C > 0$ is independent of R and y .

We are now in a position to give the proof of Theorem 1.1.

Proof. Since $w^{r'} \in A_1$, then we have $w \in A_1$ by Lemma 2.2. We now observe that for $w \in A_1$ and $p = n/(n + \alpha)$, then $[n(q_w/p - 1)] = [\alpha] = 0$. By Theorem 2.4 and Lemma 3.1, it suffices to verify that for any w -($p, r', 0$)-atom $a(x)$, there exists a constant $C > 0$ independent of a such that $\|T_\Omega(a)\|_{WL_w^p} \leq C$.

Let $a(x)$ be a w -($p, r', 0$)-atom centered at x_0 with $\text{supp } a \subseteq Q = Q(x_0, l)$, and let $Q^* = 2\sqrt{n}Q$. For any fixed $\lambda > 0$, we write

$$\begin{aligned} & \lambda^p \cdot w(\{x \in \mathbb{R}^n : |T_\Omega a(x)| > \lambda\}) \\ & \leq \lambda^p \cdot w(\{x \in Q^* : |T_\Omega a(x)| > \lambda\}) + \lambda^p \cdot w(\{x \in (Q^*)^c : |T_\Omega a(x)| > \lambda\}) \\ & = I_1 + I_2. \end{aligned}$$

Since $w \in A_1$, then by using Theorem B, we can see that T_Ω is bounded on $L_w^{r'}$. Applying Chebyshev's inequality, Hölder's inequality, Lemma 2.1 and the size condition of atom a , we thus obtain

$$\begin{aligned} I_1 & \leq \int_{Q^*} |T_\Omega a(x)|^p w(x) dx \\ & \leq \left(\int_{Q^*} |T_\Omega a(x)|^{r'} w(x) dx \right)^{p/r'} \left(\int_{Q^*} w(x) dx \right)^{1-p/r'} \\ & \leq \|T_\Omega(a)\|_{L_w^{r'}}^p w(Q^*)^{1-p/r'} \end{aligned}$$

$$\begin{aligned} &\leq C \cdot \|a\|_{L_w^{r'}}^p w(Q)^{1-p/r'} \\ &\leq C. \end{aligned} \quad (3.1)$$

We turn our attention to the estimate of I_2 . First we note that if $\{x \in (Q^*)^c : |T_\Omega a(x)| > \lambda\} = \emptyset$, then the inequality

$$I_2 \leq C$$

holds trivially. Now assume that $\{x \in (Q^*)^c : |T_\Omega a(x)| > \lambda\} \neq \emptyset$. Set $Q_1^* = Q^*$ and $Q_{k+1}^* = (Q_k^*)^*$, $k = 1, 2, \dots$. Integrating over $Q_{k+1}^* \setminus Q_k^*$ ($k \in \mathbb{Z}_+$) on both sides of the inequality $\lambda < |T_\Omega a(x)|$, then we get

$$\lambda \cdot |Q_{k+1}^* \setminus Q_k^*| \leq \int_{Q_{k+1}^* \setminus Q_k^*} |T_\Omega a(x)| dx, \quad \text{for all } k = 1, 2, \dots$$

By the cancellation condition of atom a , we have

$$\begin{aligned} \int_{Q_{k+1}^* \setminus Q_k^*} |T_\Omega a(x)| dx &= \int_{Q_{k+1}^* \setminus Q_k^*} \left| \int_Q \left(K(x, x-y) - K(x, x-x_0) \right) a(y) dy \right| dx \\ &\leq \int_Q \left\{ \int_{Q_{k+1}^* \setminus Q_k^*} |K(x, x-y) - K(x, x-x_0)| dx \right\} |a(y)| dy \end{aligned}$$

Using Hölder's inequality and Lemma 3.2, we can see that for any $y \in Q$, the inner integral of the above expression is dominated by

$$\begin{aligned} &\left(\int_{Q_{k+1}^* \setminus Q_k^*} |K(x, x-y) - K(x, x-x_0)|^r dx \right)^{1/r} \left(\int_{Q_{k+1}^* \setminus Q_k^*} 1 dx \right)^{1/r'} \\ &\leq C \cdot |Q_{k+1}^* \setminus Q_k^*|^{1/r'} (2^{kl})^{-n/r'} \left(\frac{|y-x_0|}{2^{kl}} + \int_{|y-x_0|/2^{k+1}l}^{|y-x_0|/2^{kl}} \frac{\omega_r(\delta)}{\delta} d\delta \right) \\ &\leq C \cdot |Q_{k+1}^* \setminus Q_k^*|^{1/r'} (2^{kl})^{-n/r'} \left(\frac{1}{2^k} + \int_{|y-x_0|/2^{k+1}l}^{|y-x_0|/2^{kl}} \frac{\omega_r(\delta)}{\delta} d\delta \right) \end{aligned} \quad (3.2)$$

On the other hand, it follows directly from Hölder's inequality, the A_r condition and the size condition of atom a that

$$\begin{aligned} \int_Q |a(y)| dy &\leq \left(\int_Q |a(y)|^{r'} w(y) dy \right)^{1/r'} \left(\int_Q w(y)^{-r/r'} dy \right)^{1/r} \\ &\leq C \cdot \|a\|_{L_w^{r'}} \frac{|Q|}{w(Q)^{1/r'}} \\ &\leq C \cdot \frac{|Q|}{w(Q)^{1/p}}. \end{aligned}$$

In addition, since $Q \subseteq Q_k^*$ for any $k = 1, 2, \dots$, then by Lemma 2.3, we can get

$$\frac{w(Q)}{w(Q_k^*)} \geq C \cdot \frac{|Q|}{|Q_k^*|},$$

which implies

$$\int_Q |a(y)| dy \leq C \cdot \left(\frac{|Q_k^*|}{|Q|} \right)^{1/p-1} \frac{|Q_k^*|}{w(Q_k^*)^{1/p}}. \quad (3.3)$$

Summarizing the estimates (3.2) and (3.3) derived above and using the fact that $p = n/(n + \alpha)$, we then obtain

$$\begin{aligned} & \int_{Q_{k+1}^* \setminus Q_k^*} |T_\Omega a(x)| dx \\ & \leq C \cdot \frac{|Q_{k+1}^* \setminus Q_k^*|^{1/r'} |Q_k^*|^{1/r}}{w(Q_k^*)^{1/p}} \left(\frac{|Q_k^*|}{|Q|} \right)^{\alpha/n} \left(\frac{1}{2^k} + \int_{|y-x_0|/2^{k+1}l}^{|y-x_0|/2^k l} \frac{\omega_r(\delta)}{\delta} d\delta \right) \end{aligned}$$

It is obvious that for any $k = 1, 2, \dots$, we have $|Q_{k+1}^* \setminus Q_k^*| \geq |Q_k^*|$. Hence

$$\begin{aligned} \lambda & < \frac{1}{|Q_{k+1}^* \setminus Q_k^*|} \cdot \int_{Q_{k+1}^* \setminus Q_k^*} |T_\Omega a(x)| dx \\ & \leq C \cdot \frac{|Q_k^*|^{1/r}}{|Q_{k+1}^* \setminus Q_k^*|^{1/r}} \frac{1}{w(Q_k^*)^{1/p}} \cdot 2^{k\alpha} \left(\frac{1}{2^k} + \frac{1}{2^{k\alpha}} \int_0^1 \frac{\omega_r(\delta)}{\delta^{1+\alpha}} d\delta \right) \\ & \leq C \cdot \left(1 + \int_0^1 \frac{\omega_r(\delta)}{\delta^{1+\alpha}} d\delta \right) \frac{1}{w(Q_k^*)^{1/p}}. \end{aligned}$$

Furthermore, for $p = n/(n + \alpha)$, it is easy to check that

$$\lim_{k \rightarrow \infty} \frac{1}{w(Q_k^*)^{1/p}} = 0.$$

Then for any given $\lambda > 0$, we are able to find a maximal positive integer \mathcal{K} such that

$$\lambda < C \cdot \frac{1}{w(Q_{\mathcal{K}}^*)^{1/p}}.$$

Therefore

$$\begin{aligned} I_2 & \leq \lambda^p \cdot \sum_{k=1}^{\mathcal{K}} w(\{x \in Q_{k+1}^* \setminus Q_k^* : |Ta(x)| > \lambda\}) \\ & \leq C \cdot \frac{1}{w(Q_{\mathcal{K}}^*)} \sum_{k=1}^{\mathcal{K}} w(Q_{k+1}^*) \\ & \leq C. \end{aligned} \quad (3.4)$$

Combining the above inequality (3.4) with (3.1) and taking the supremum over all $\lambda > 0$, we conclude the proof of Theorem 1.1. \square

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